THE DEVIANT DYNAMICS OF DEATH IN HETEROGENEOUS POPULATIONS

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FOREWORD

Low fertility levels in IIASA countries are creating aging populations whose demands for health care and income maintenance (social security) will increase to unprecedented levels, thereby calling forth policies that will seek to promote increased family care and worklife flexibility. The Population Program is examining current patterns of population aging and changing lifestyles in IIASA countries, projecting the needs for health and income support that such patterns are likely to generate during the next several decades, and considering alternative family and employment policies that might reduce the social costs of meeting these needs.

The program is seeking to develop a better understanding of how low fertility and mortality combine to create aging populations, with high demands for health and income maintenance, and reduced family support systems that can provide that maintenance. The research will produce analyses of current demographic patterns in IIASA countries together with an assessment of their probable future societal consequences and impacts on the aging. It will consider the position of the elderly within changing family structures, review national policies that seek to promote an enlarged role for family care, and examine the costs and benefits of alternative systems for promoting worklife flexibility by transferring income between different periods of life.

In this report, James Vaupel (USA) and Anatoli Yashin (USSR) examine the impacts of heterogeneity on populations whose members are gradually making some major transition. Their focus is on human mortality, but the mathematics they develop is relevant to studies of, for example, migration, morbidity, marriage, criminal recidivism, drug addiction, and the reliability of equipment. The authors show that the observed dynamics of the surviving population – the population that has not yet made the transition – will systematically deviate from the dynamics of the behavior of any of the individuals that make up the aggregate population. Furthermore, they develop methods for uncovering the underlying dynamics of individual behavior, given observations of population behavior. These methods will be useful in explaining and predicting demographic patterns. In addition, because the impact of a policy intervention can sometimes only be correctly predicted if the varying responses of different kinds of individuals are taken into account, the methods should prove to be of value to policy analysts.

A list of related IIASA publications appears at the end of this report.

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SUMMARY

The members of most populations gradually die off or drop out: people die, machines wear out, residents move out, etc. In many such "aging" populations, some members are more likely to "die" than others. Standard analytical methods largely ignore this heterogeneity; the methods assume that all members of a population cohort at a given age face the same probability of death. This paper presents some mathematical methods for studying how the behavior over time of a heterogeneous cohort deviates from the behavior of the individuals that make up the cohort. The methods yield some startling results: individuals age faster than cohorts, eliminating a cause of death can decrease *life expectancy, a cohort can suffer a higher death rate than another cohort even though its members have lower death rates, and cohort death rates can be increasing even though its members' death rates are decreasing.*

WHAT DIFFERENCE DO DIFFERENCES MAKE?

Many systems are aggregations of similar objects. Forests are collections of trees; flocks are congregations of birds or sheep; cities are amalgams of buildings; plants and animals are built up of cells. The units in such aggregations usually have limited life spans and evolve and change over their life before they die or are renewed. The units, although similar, are rarely identical; even two mass-produced automobiles of the same make and model can differ substantially. In studying populations of similar objects, however, and in analyzing the impact of interventions and control policies, the simplifying assumption is often made that the units are identical. A key question thus is: what difference does it make to ignore individual differences and to treat a population as homogeneous when it is actually heterogeneous?

This report examines some aspects of this question. The focus is on patterns over time in aging and life-cycle processes and, more specifically, on jumps and transitions in these processes. Examples abound. Animals and plants die, the healthy fall ill, the unemployed find jobs, the childless reproduce, and the married divorce. Residents move out,

machines wear out, natural resources get used up, and buildings are torn down. Infidels convert, ex-convicts recidivate, abstainers become addicted, and hold-outs adopt new technologies. Regularities in these processes are studied by researchers in such diverse specialties as reliability and maintenance engineering, epidemiology, health care planning, actuarial statistics, and criminology, as well as by analysts in disciplines such as demography, economics, ecology, sociology, and policy analysis.

In many collections or populations, some units are more likely to make a transition than others. Standard analytical methods largely ignore this heterogeneity; the methods assume that all members of a population (or subpopulation, such as US black males) at a given age face the same probability of change. This paper presents some methods for studying what difference heterogeneity within a population makes in the behavior of a changing population over time.

The analytical methods will be illustrated by examples drawn from the study of human mortality, and, henceforth, the word "death" will be used instead of the more general terms "change" and "transition". Readers interested in areas of applications other than human mortality should associate death with a more appropriate analogous word like failure, separation, occurrence, or movement.

The focus on human mortality implies a focus on the simplest kind of life-cycle process, i.e., a process with just one transition that leads to exit. This simplicity permits the effects of heterogeneity to be clearly shown and readily explained. The focus on human mortality gives the exposition a concreteness that fosters intelligibility. Furthermore, it turns out that the analytical methods yield some stimulating insights and policy implications when applied to human mortality.

ROOTS OF THE RESEARCH

A small but growing body of research is relevant to the analysis of differences in behavior over time between heterogeneous and homogeneous populations. Some strands of this research can be traced back to Cournot's study of judicial decisions (1838) and Weinberg's investigation of the frequency of multiple births (1902). Greenwood and Yule's analysis of differences in accident proneness and susceptibility to illness (1920) was followed up by Lundberg (1940), Arbous and Kerrich (1951), and Cohen and Singer (1979). Gini (1924) considered heterogeneity in female fecundity; Potter and Parker (1964) and Sheps and Menken (1973) developed this approach. In their influential study of the industrial mobility of labor, Blumen, Kogan, and McCarthy (1955) distinguished "movers" from "stayers" and then considered an arbitrary number of groups with different "proneness to movement"; Silcock (1954) used a continuous distribution over individuals to describe the "rate of wastage" in labor turnover. This research on the mobility of labor was generalized and extended to such related fields as income dynamics and geographic migration by Spilerman (1972), Ginsberg (1973), Singer and Spilerman (1974), Kitsul and Philipov (1981), and Heckman and Singer (1982), among others. Harris and Singpurwalla (1968) and Mann, Schafer, and Singpurwalla (1974) developed methods for taking into account differences in reliability among machines and equipment. Shepard and Zeckhauser (1975, 1977, 1980a,b; Zeckhauser and Shepard 1976) pioneered the analyses of heterogeneity in human mortality and morbidity; Woodbury and Manton (1977), Keyfitz and

Littman (1980), Manton and Stallard (1979, 1981a,b), and Vaupel, Manton, and Stallard (1979a; Manton *et al.* 1981) have made further contributions.

This rich body of research indicates that there is a core of mathematical methods that can be usefully applied to the analysis of heterogeneity in such diverse phenomena as accidents, illness, death, fecundity, labor turnover, migration, and equipment failure. These sundry applications and the varied disciplinary backgrounds of the researchers make it hardly surprising that key elements of this common core of mathematics were independently discovered by several researchers. Further progress, however, surely would be accelerated if the wide applicability of the underlying mathematics of heterogeneity were recognized.

A UNIFYING QUESTION

Building on this body of research and, most directly, on Vaupel *et al.* (1979a), this report addresses a basic question: how does the observed rate of death, over time, for a cohort of individuals born at the same time relate to the probability of death, over time, for each of the individuals in the cohort.* This question provides a unifying focus for developing the mathematical theory of the dynamics of heterogeneous populations. It is also a useful question in applied work because researchers usually observe population death rates but often are interested in individual death rates, for three main reasons. First, the effect of a policy or intervention may depend on individual responses and behavior. Second, individual rates may follow simpler patterns than the composite population rates. And third, explanation of past rates and prediction of future rates may be improved by considering changes on the individual level.

It turns out that the deviation of individual death rates from population rates implies some surprising and intriguing results. Individuals "age" faster than heterogeneous cohorts. Eliminating a cause of death can *decrease* subsequent observed life expectancy. A population can suffer a higher death rate at older ages than another population even though its members have lower death rates at all ages. A population's death rate can be increasing even though its members' death rates are decreasing.

The theory leads to some methods that may be of use to policy analysts in evaluating the effects of various interventions, e.g., a medical care program that reduces mortality rates at certain ages. Shepard and Zeckhauser (1980b) develop and discuss some methods of this kind. The theory also yields predictions that may be of considerable interest to policy analysts. For example, in the developed countries of the world, death rates after age 70 and especially after age 80 may decline faster – and at an accelerating rate – than

*The word "rate" means different things to different specialists. In this report, "rate of death" is a measure of the likelihood of death at some instant. As noted later in the text, the phrase "rate of death" as used here, has numerous aliases, including hazard rate and force of mortality. The "rate of death for an individual" or "individual death rate" is defined by equation (1a); the "cohort death rate" is defined by equation (1b). Note that "rate of death", as used here, is neither a probability nor an average over some time period. Furthermore, note that the rate of death for an individual is a function of that individual's probability of death at some instantaneous age conditional on the individual's surviving to that age. Some readers may find it helpful to mentally substitute "force of mortality" for "rate of death" whenever the phrase appears.

now predicted by various census and actuarial projections. As a result, pressures on social security and pension systems may be substantially greater than expected.

MATHEMATICAL PRELIMINARIES

Let Ω be some set of parameters ω. Assume that each parameter value characterizes a homogeneous class of individuals and that the population is a mix of these homogeneous classes in proportions given by some probability distribution on Ω.

Denote by $p_\omega(x)$ the probability that an individual from homogeneous class ω will be alive at age x, and let $\mu_\omega(x)$ be the instantaneous age-specific death rate at age x for an individual in class ω. By definition,

$$\mu_\omega(x) = -[dp_\omega(x)/dx]/p_\omega(x) \tag{1a}$$

Similarly, let $\bar{p}(x)$ be the probability that an arbitrary individual from the population will be alive at age x. That is, let $\bar{p}(x)$ be the expected value of the probability of surviving to age x for a randomly chosen individual at birth. Alternatively, $\bar{p}(x)$ can be interpreted as the expected value of the proportion of the birth cohort that will be alive at age x. The cohort death rate $\bar{\mu}(x)$ is then defined by

$$\bar{\mu}(x) = -[d\bar{p}(x)/dx]/\bar{p}(x) \tag{1b}$$

Throughout this paper, superscript bars will be used to denote variables pertaining to expected values either for a randomly chosen individual at birth or, equivalently, for the entire cohort.

Suppose that all the individuals in a population were identical and that their chances of survival were described by $p(x)$. Then, it turns out that $\bar{p}(x)$ would be the same as $p(x)$. Thus, a cohort described by $\bar{p}(x)$ could be interpreted as being a homogeneous population comprised of identical individuals each of whom had life-chances given by $p(x)$ equaling $\bar{p}(x)$. This remarkable fact means that researchers interested in population rates can simplify their analysis by ignoring heterogeneity; this simplification has permitted the development of demography, actuarial statistics, reliability engineering, and epidemiology.

For some purposes, however, the simplification is inadequate, counter-productive, or misleading. For example, sometimes researchers are interested in individual rather than population behavior, sometimes patterns on the individual level are simpler than patterns on the population level, and sometimes the impact of a policy intervention can only be correctly predicted if the varying responses of different kinds of individuals are taken into account. That is, sometimes individual differences make enough difference that it pays to pay attention to them; a variety of specific examples are given later in this report. Furthermore, the complexities introduced by heterogeneity are not intractable; indeed, the mathematical methods presented in this paper are fairly simple.

The expected proportion of the entire population that is alive at time x and that will die in the period from x to $x + 1$ is given by the formula

$$\bar{q}(x) = 1 - \exp\left[-\int_x^{x+1} \bar{\mu}(t)\,dt\right] \tag{2a}$$

When $\bar{\mu}(x)$ is small and does not change significantly in the period from x to $x + 1$, then

$$\bar{q}(x) \approx \bar{\mu}(x) \tag{2b}$$

Consequently, $\bar{\mu}(x)$ is often intuitively interpreted as describing the probability of death.

Because of their instantaneous nature, death rates like $\bar{\mu}(x)$ and $\mu_\omega(x)$ are often more mathematically convenient than probabilities like $\bar{q}(x)$ or other statistics such as life expectancy or life-span fractiles; the mathematical methods of this report will be derived largely in terms of death rates. As might be expected, the rate of death is commonly used in various applications and has numerous aliases, including hazard rate, mortality rate, failure rate, occurrence rate, transition rate, rate of wastage, force of mortality, force of separation, force of mobility, conditional risk, death intensity, transition intensity, intensity of migration, and intensity of risk.

BASIC MATHEMATICAL FORMULATION

In mortality analysis, the adjective "heterogeneous" usually implies that individuals of the same age differ in their chances of death. As in many other problems involving relative measurement, it is useful to have some standard or baseline to which the death rates of various individuals can be compared. Let $\mu(x)$ be this standard, baseline death rate; how values of $\mu(x)$ might be chosen will be discussed later. The "relative-risk" for individuals in homogeneous class ω at time x will be defined as

$$z(x,\omega) = \mu_\omega(x)/\mu(x) \tag{3}$$

It is convenient to use $\mu(x,z)$ to denote the death rate at time x of individuals at relative-risk z. Clearly,

$$\mu(x,z) = z\mu(x) \tag{4}$$

Thus,

$$\mu(x) = \mu(x,1) \tag{5}$$

The standard death rate $\mu(x)$ can therefore be interpreted as the death rate for the class of individuals who face a relative-risk of one.

This formulation is simple and broadly applicable. More importantly, it yields a powerful result that is central to the mathematics of heterogeneity. Let $f_x(z)$ denote the conditional density of relative-risk among survivors at time x. As shown in the Appendix, the expected death rate in the population $\bar{\mu}(x)$ is the weighted average of the death rates of the individuals who comprise the population:

$$\bar{\mu}(x) = \int_0^\infty \mu(x,z) f_x(z)\, dz \tag{6}$$

Since $\bar{z}(x)$, the mean of the relative-risk values of time x, is given by

$$\bar{z}(x) = \int_0^\infty z\, f_x(z)\, dz \tag{7}$$

it follows from equation (4) that

$$\bar{\mu}(x) = \mu(x)\bar{z}(x) \tag{8}$$

This simple result is the fundamental theorem of the mathematics of heterogeneity, since it relates the death rate for the population to the death rates for individuals. The value of $\mu(x)$ gives the death rate for the hypothetical "standard" individual facing a relative-risk of one; multiplying $\mu(x)$ by z gives the death rate for an individual facing a relative-risk of z. The value of $\bar{z}(x)$ gives the average relative-risk of the surviving population at time x. In interpreting this it may be useful, following Vaupel *et al.* (1979a), to view z as a measure of "frailty" or "susceptibility". Thus, $\bar{z}(x)$ measures the average frailty of the surviving cohort.

UNCHANGING FRAILTY

The relationship over time of $\bar{\mu}(x)$ versus $\mu(x)$ is determined by the trajectory of $\bar{z}(x)$. The simplest case to study is the case where individuals are born at some level of relative-risk (or frailty) and remain at this level all their lives. In this case, the only factor operating to change $\bar{z}(x)$ is the higher mortality of individuals at higher levels of relative-risk; thus, this pure case most clearly reveals the effects of differential selection and the survival of the fittest. Although most of this report addresses this special case, some generalizations are discussed later. Because the mathematics derived for the special case also holds for a broader range of assumptions, the special case is less restrictive than it may seem at first.

Imagine a population cohort that is born at some point in time. Let $f_0(z)$ describe the proportion of individuals in the population born at various levels of relative-risk z; $f_0(z)$ can be interpreted as a probability density function. Assume that each individual remains at the same level of z for life. For convenience, the mean value of $f_0(z)$ might as well be taken as one, so that the standard individual at relative-risk one is also the mean individual at birth and so that $\mu(0)$ equals $\bar{\mu}(0)$. As before, let $\mu(x,z)$ and $\mu(x)$ be the death rates of individuals at relative-risk z and of the standard individual. Let $H(x,z)$ be the cumulative "hazard" experienced from birth to time x:

$$H(x,z) = \int_0^x \mu(x,z)\, dx \tag{9}$$

Clearly,

$$H(x,z) = z\, H(x) \tag{10}$$

The probability that an individual at relative-risk z will survive to age x is given by

$$p(x,z) = p(x)^z = \exp\,[-z\,H(x)] \tag{11}$$

Consequently,

$$f_x(z) = \frac{f_0(z)\exp\,[-z\,H(x)]}{\int_0^\infty f_0(z)\exp\,[-z\,H(x)]\;\mathrm{d}z} \tag{12}$$

where the denominator is a scaling factor equal to $\bar{p}(x)$, the proportion of the population cohort that has survived to age x. Thus

$$\bar{z}(x) = \frac{\int_0^\infty z f_0(z)\exp\,[-z\,H(x)]\;\mathrm{d}z}{\int_0^\infty f_0(z)\exp\,[-z\,H(x)]\;\mathrm{d}z} \tag{13}$$

Differentiating equation (13) with respect to x yields

$$\mathrm{d}\bar{z}(x)/\mathrm{d}x = -\mu(x)\,\sigma_z^2(x) \tag{14}$$

where $\sigma_z^2(x)$ is the conditional variance of z among the population that is alive at time x. Since $\mu(x) > 0$ and $\sigma_z^2(x) > 0$, the value of $\mathrm{d}\bar{z}(x)/\mathrm{d}x$ must be negative. Therefore, as might be expected, the mean relative-risk declines over time as death selectively removes the frailest members of the population. This means that $\mu(x)$ increases more rapidly than $\bar{\mu}(x)$: individuals "age" faster than heterogeneous cohorts.

If $\mu(x)$ is greater than zero for all x, then

$$\bar{z}(x) > \bar{z}(x') \text{ iff } x < x' \tag{15a}$$

and

$$\bar{p}(x) < \bar{p}(x') \text{ iff } x < x' \tag{16}$$

Consequently,

$$\bar{z}\,[\bar{p}^{-1}(\bar{p})] < \bar{z}\,[\bar{p}^{-1}(\bar{p}')] \text{ iff } \bar{p} < \bar{p}' \tag{15b}$$

where $\bar{p}^{-1}(\bar{p})$ is the inverse function of $\bar{p}(x)$, and $\bar{p}$ and $\bar{p}'$ are two specific values of the survival function. That is, mean relative-risk declines monotonically not only with age (or time) x but also with the proportion surviving $\bar{p}$.

HOW μ DIVERGES FROM $\bar{\mu}$

The *magnitude* of the divergence of $\mu(x)$ from $\bar{\mu}(x)$ depends on the distribution of relative-risk. Several researchers in different fields, including Silcock (1954), Spilerman (1972), Mann, Schafer, and Singpurwalla (1974), and Vaupel, Manton, and Stallard (1979a), have discovered that the gamma distribution is especially convenient to work with, since it is one of the best known nonnegative distributions, is analytically tractable, and takes on a variety of shapes depending on parameter values. If the mean relative-risk at birth is one, then the gamma probability density function at birth is given by

$$f_0(z) = k^k z^{k-1} \exp(-kz)/\Gamma(k) \tag{17}$$

where k, the so-called shape parameter, equals (when the mean is one) the inverse of the variance σ^2. When k equals one, the distribution is identical to the exponential distribution; when k is large, the distribution assumes a bell-shaped form reminiscent of a normal distribution.

If relative-risk at birth is gamma distributed with mean one, it can be shown (see Vaupel *et al.* 1979a) that

$$\bar{z}(x) = \bar{p}(x)^{\sigma^2} = \exp[-\sigma^2 \bar{H}(x)] = \exp\left[-\sigma^2 \int_0^x \bar{\mu}(x)\,dx\right] \tag{18}$$

and that

$$\bar{z}(x) = 1/[1 + \sigma^2 H(x)] \tag{19}$$

Thus, the relationship of $\mu(x)$ to $\bar{\mu}(x)$, as determined by $\bar{z}(x)$, can be determined by the cumulative hazard for either the population or the standard individual. In the special case where σ^2 equals one, the value of $\bar{z}(x)$ falls off with $\bar{p}(x)$, the proportion of the cohort that is surviving. It also can be shown (Vaupel *et al.* 1979a) that $f_x(z)$ is gamma distributed, with a mean of $\bar{z}(x)$ and a shape parameter equal to the same value of k as at birth.

These results for the gamma distribution with mean one at birth are easily generalized to the case of any mean $\bar{z}(0)$ at birth. Equation (18) then becomes

$$\bar{z}(x) = \bar{z}(0)\bar{p}(x)^{\sigma^2} \tag{18'}$$

and equation (19) becomes

$$\bar{z}(x) = \bar{z}(0)/[1 + \sigma^2 \bar{z}(0) H(x)] \tag{19'}$$

There is, however, little reason to use this generalized formulation. Let

$$\bar{z}'(x) = \bar{z}(x)/\bar{z}(0) \tag{20a}$$

and

$$\mu'(x) = \mu(x)/z(0) \tag{20b}$$

This simple and harmless transformation converts equations (18′) and (19′) back to (18) and (19). Furthermore, as indicated earlier, the standard death rate $\mu(x)$ might as well be associated with the mean individual at birth.

Instead of working with a gamma distribution, it might seem more natural to assume that there is some normally distributed risk factor w that determines relative-risk z:

$$z = w^2 \tag{21}$$

It turns out that if w is normally distributed with mean zero and any variance σ^2, then z will be gamma distributed with a shape parameter of one-half. Thus nothing is to be gained by working with the normal distribution with mean zero rather than with a gamma distribution.

In the "mover/stayer" model developed by Blumen *et al.* (1955), individuals fall into two groups with relative-risk z_1 and z_2. The value of z_1 can be assumed equal to zero, but more generally z_1 can simply be taken as less than z_2. Using equation (13), it is not difficult to confirm that when mean relative-risk at birth is one,

$$\bar{z}(x) = \frac{[z_1/(1-z_1)]\exp[-z_1H(x)] + [z_2/(z_2-1)]\exp[-z_2H(x)]}{[1/(1-z_1)]\exp[-z_1H(x)] + [1/(z_2-1)]\exp[-z_2H(x)]} \tag{22}$$

Consequently, $\bar{z}(x)$ will start at a value of one when x is zero and will fall off to a value of z_1 as the individuals at relative-risk z_2 die off at a relatively rapid rate.

Another distribution of interest may be the uniform distribution, stretching from $1-\alpha$ to $1+\alpha$, with $\alpha \leqslant 1$. In this case, it is possible to show that

$$\bar{z}(x) = 1 - \alpha\left(\frac{\exp[\alpha H(x)] + \exp[-\alpha H(x)]}{\exp[\alpha H(x)] - \exp[-\alpha H(x)]}\right) + \frac{1}{H(x)} \tag{23}$$

In deriving this result, it is helpful to realize that $\bar{z}(x)$ can be considered to be a function of H and that the equation for $\bar{z}(H)$ can be expressed as

$$\bar{z}(H) = [\mathrm{d}f^*(H)/\mathrm{d}H]/f^*(H) \tag{24}$$

where $f^*(H)$ is the Laplace transform of $f_0(z)$. Equation (23) implies that $\bar{z}(x)$ approaches $1-\alpha$ as x increases.

Although formulas for $\bar{z}(x)$ have not been derived for other distributions, the value of $\bar{z}(x)$ can generally be readily computed, to a close approximation, by applying numerical methods to equation (13). The values in Table 1 for the Weibull and lognormal distributions were calculated in this way.

TABLE 1 The divergence of μ from $\bar{\mu}$.

Variance and forms of initial distribution of relative-risk	Values of $\mu/\bar{\mu}$ when $\bar{p}$ is:					
	1.00	0.75	0.50	0.25	0.10	0.05
$\sigma^2 = 0.1$						
Gamma	1.00	1.03	1.07	1.15	1.26	1.35
Weibull	1.00	1.03	1.08	1.17	1.34	1.49
Lognormal	1.00	1.03	1.07	1.14	1.23	1.30
$\sigma^2 = 1$						
Exponential[a]	1.00	1.33	2.00	4.00	10.00	20.00
Lognormal	1.00	1.27	1.64	2.30	3.33	4.24
$\sigma^2 = 2$						
Gamma	1.00	1.78	4.00	16.00	100.00	400.00
Weibull	1.00	1.70	3.32	9.56	36.10	99.01
Lognormal	1.00	1.49	2.23	3.46	5.61	7.65

[a]When $\sigma^2 = 1$, the gamma and Weibull distributions are identical to the exponential distribution.

Table 1 is designed to show how $\mu(x)$ diverges from $\bar{\mu}(x)$ given different initial distributions of relative-risk with different variances. The table presents values of $\mu(x)$ divided by $\bar{\mu}(x)$, which equals the inverse of $\bar{z}(x)$. The results are presented for different values of $\bar{p}(x)$, the proportion of the intial population that is surviving; presenting the results for values of $\bar{p}(x)$ rather than for values of x is convenient since assumptions about the rate of aging over time (i.e., about how $\mu(x)$ changes with x) do not have to be made. The table indicates that $\mu(x)$ can be substantially greater than $\bar{\mu}(x)$ when only a fraction of the population is alive. Even when the variance in relative-risk is only 0.1 (compared with a mean level at birth of one), $\mu(x)$ is 30 to 50 percent higher than $\bar{\mu}(x)$ when 5 percent of the population is surviving. As the table demonstrates, the degree of divergence of $\mu(x)$ from $\bar{\mu}(x)$ depends on both the form of the initial distribution of relative-risk and the variance of this distribution.

THE SHAPE OF THE AGING TRAJECTORY

Although Table 1 and equations (18), (19), (22), and (23) provide information about the amount of divergence between $\mu(x)$ and $\bar{\mu}(x)$, analysis of the *shape* of $\mu(x)$ and $\bar{\mu}(x)$ requires some assumptions about how one of these two curves increases with x. If relative-risk at birth is gamma distributed with mean one and variance σ^2, then the correspondence between six different formulas for $\mu(x)$ and $\bar{\mu}(x)$ is given in Table 2. Figure 1 depicts how the curves for $\mu(x)$ and $\bar{\mu}(x)$ diverge in four cases. The table and figure clearly demonstrate that the pattern of individual aging can radically differ from the observed pattern of aging in the surviving cohort. For instance, when $\mu(x)$ is constant, $\bar{\mu}(x)$ declines with age; heterogeneity introduces spurious age-dependence on the population level.

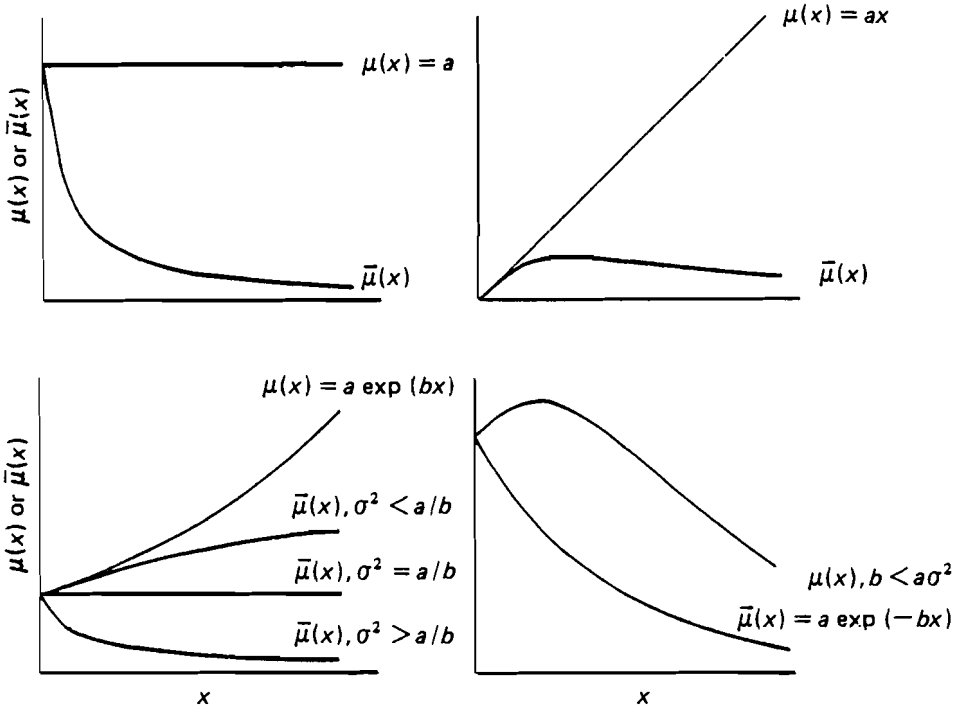

FIGURE 1 Patterns of divergence. The examples depict the trajectories of $\mu(x)$ and $\bar{\mu}(x)$ that correspond to some of the algebraic expressions presented in Table 2.

TABLE 2 Individuals age faster than heterogeneous cohorts.

When z is gamma distributed with variance σ^2 at birth:	
If the value of $\mu(x)$ is given by:	...then the value of $\bar{\mu}(x)$ is given by:
a	$a/(1 + \sigma^2 ax)$
ax	$ax/(1 + \sigma^2 ax^2/2)$
$a \exp(bx)$	$a \exp(bx)/\{1 + \sigma^2 a[\exp(bx) - 1]/b\}$
$a \exp(-bx)$	$a \exp(-bx)/\{1 + \sigma^2 a[1 - \exp(-bx)]/b\}$
$a \exp(bx) \exp\{\sigma^2 a[\exp(bx) - 1]/b\}$	$a \exp(bx)$
$a \exp(-bx) \exp\{\sigma^2 a[1 - \exp(-bx)]/b\}$	$a \exp(-bx)$

NOTES: If $\mu(x) = ax$, then $\bar{\mu}(x)$ reaches a maximum of $(a/2\sigma^2)^{1/2}$ when $x = (2/a\sigma^2)^{1/2}$. If $\mu(x) = a \exp(bx)$, then as $x \to \infty$, $\bar{\mu}(x) \to b/\sigma^2$. If $\bar{\mu}(x) = a \exp(bx)$ (i.e., follows a Gompertz curve), then the ratio of $\mu(x)$ to $\bar{\mu}(x)$ can be expressed as a double-exponential equation: $\mu(x)/\bar{\mu}(x) = \alpha\beta^{\gamma^x}$.

THE DISTRIBUTION OF LIFE SPANS

Although the discussion so far has focused on the divergence of μ and $\bar{\mu}$ over time, comparisons of individual versus cohort behavior in heterogeneous populations could also be expressed in terms of other statistics. Consider, for example, the fractiles of the distribution of life spans or, equivalently, the distribution of age of death. Table 3 presents

some of these fractiles for a population and for individuals. Fractiles for the standard individual are given for three levels of heterogeneity, as measured by σ^2; fractiles are also presented for individuals at three levels of relative-risk z. The calculations assume that relative-risk is gamma distributed with mean one at birth and that the observed death rate for the population is given by a Gompertz function, $a \exp(bx)$, where a equals 0.00012 and b equals 0.085. The table indicates that the distribution of life spans in a population is more spread out than the distribution of possible life spans for an individual. In particular, the right-hand tail of the distribution is shorter for individuals, especially for robust individuals where variance in heterogeneity is high.

TABLE 3 The distribution of life spans.

Category	Age at which the probability of being alive equals: 0.75	0.50	0.25	0.10	0.01	0.001	Length of right-hand tail $x_{0.001} - x_{0.50}$
For entire cohort	62.6	72.9	81.1	87.0	95.2	100.0	27.1
For individuals							
$z = 1$							
$\sigma^2 = 0.1$	62.4	72.5	80.3	85.8	92.9	96.7	24.2
$\sigma^2 = 1$	61.1	69.7	75.6	79.3	83.6	85.8	16.1
$\sigma^2 = 10$	53.8	58.8	61.9	63.8	66.0	67.2	8.4
$\sigma^2 = 1$							
$z = 0.1$	80.8	85.8	88.9	90.8	93.1	94.2	8.4
$z = 1$	61.1	69.7	75.6	79.3	83.6	85.8	16.1
$z = 10$	35.9	45.7	53.3	58.8	65.8	69.7	24.0

NOTE: See text for discussion and explanation of underlying assumptions.

MORTALITY CONVERGENCE AND CROSSOVER

For many pairs of populations, mortality rates converge and even cross over with age. For example, blacks in the United States have lower mortality rates than US whites after age 75 or so (Shepard and Zeckhauser 1980b, Manton and Stallard 1981a). In 1980, Puerto Ricans had a longer life expectancy at age 65 than the residents of any other country or area for which statistics were available (Vaupel 1978). In most developed countries, male and female death rates converge in old age. Nam *et al.* (1978) present statistics on this and a variety of other convergences and crossovers.

These convergences and crossovers of population death rates may be artifacts of heterogeneity in individual death rates. Let $r(x)$ denote the ratio of death rates for the standard individual in population 2 versus 1:

$$r(x) = \mu_2(x)/\mu_1(x) \tag{25a}$$

Similarly, let $\bar{r}(x)$ denote the ratio of the population death rates:

$$\bar{r}(x) = \bar{\mu}_2(x)/\bar{\mu}_1(x) \tag{25b}$$

For simplicity, assume that the ratio is constant over time on the individual level, so that individuals at any level of relative-risk in the second population are always r times more likely to die than corresponding individuals in the first population:

$$r(x) = r > 1 \qquad \text{for all } x \tag{26}$$

Further assume that relative-risk is gamma distributed in the two populations with mean one and variances σ_1^2 and σ_2^2 at birth. Let

$$\rho = \sigma_2^2 / \sigma_1^2 \tag{27}$$

Then it follows from equations (8) and (19) that at birth

$$\bar{r}(0) = r(0) = r \tag{28a}$$

but as x increases

$$\bar{r}(x) \to 1/\rho \tag{28b}$$

Depending on the value of ρ, i.e., on the ratio of the variances in relative-risk, $\bar{r}(x)$ can either increase or decrease. If ρ is greather than one, $\bar{r}(x)$ will fall to a value less than one. This means that although, on the individual level, $\mu_2(x)$ is always r times higher than $\mu_1(x)$, the cohort death rate $\bar{\mu}_2(x)$ will start out higher than $\bar{\mu}_1(x)$ and will end up below $\bar{\mu}_1(x)$. The crossover point will occur when

$$\bar{p}_1(x) = (\rho - 1)/(\rho - 1/r) \tag{29}$$

where $\bar{p}_1(x)$ is the proportion of population 1 still surviving at age x. For example, if r equals 2 and ρ equals 1.5, the crossover will occur when $\bar{p}_1(x)$ equals 0.5. Figure 2 compares the trajectories of r and $\bar{r}$; Table 4 presents some specific numerical results.

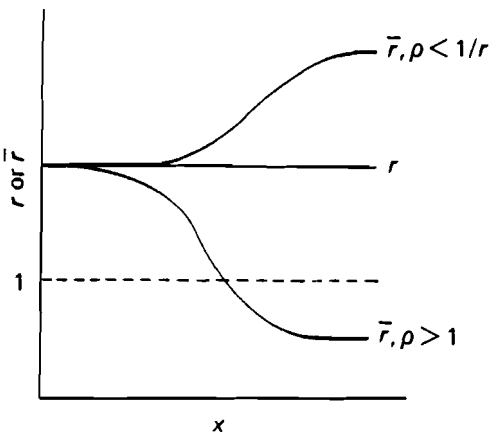

FIGURE 2 Patterns of mortality convergence and divergence.

TABLE 4 A mortality crossover.

x	$\bar{\mu}_1$	$\bar{\mu}_2$	$\bar{r} = \bar{\mu}_2/\bar{\mu}_1$	μ_1	μ_2	$r = \mu_2/\mu_1$
0	0.00010	0.00020	2.00	0.00010	0.00020	2.0
20	0.00073	0.00144	1.96	0.00074	0.00148	2.0
40	0.00518	0.00899	1.74	0.00546	0.01092	2.0
60	0.02877	0.03092	1.07	0.04034	0.08069	2.0
70	0.05233	0.04075	0.78	0.10966	0.21933	2.0
90	0.08902	0.04851	0.54	0.81031	1.62062	2.0
∞	–	–	–	–	–	2.0

ASSUMPTIONS: In calculating this table, which illustrates how an observed crossover in death rates in two populations may be an artifact of heterogeneity, it was assumed that relative-risk is gamma distributed with mean one, $\sigma_1^2 = 1$, and $\sigma_2^2 = 2$. Furthermore $\mu_2(x) = 2\mu_1(x)$, for all x, where $\mu_1(x) = 0.0001 \exp(0.1x)$.

Empirical data on convergences and crossovers in mortality rates can be used to estimate the degree of heterogeneity in relative-risk in a population. If some assumption is made about the distribution of relative-risk (e.g., that it is gamma distributed) and about the relationship of $\mu_1(x)$ to $\mu_2(x)$ (e.g., that one is a constant multiple of the other), then estimates of the variance in heterogeneity can be calculated. Vaupel *et al.* (1979b) and Manton *et al.* (1981) applied this method to various cohorts of the four populations of male and female Swedes and US whites. The results suggest that for these populations, the variance in heterogeneity is roughly one.

GERONTOLOGICAL FAILURES OF PEDIATRIC SUCCESS

Heterogeneity slows observed rates of progress in reducing population death rates at older ages. Essentially, reductions in death rates at younger ages permit frailer individuals to survive to older ages. This influx of frailer individuals serves as a brake or countercurrent on reductions in mortality rates at the older ages; Vaupel *et al.* (1979a) and Shepard and Zeckhauser (1980b) recognize this.

As a simple illustration, divide life into two parts – youth and old age, say – at age x_0. Suppose that a proportion $\bar{p}(x_0)$ of each birth cohort used to survive to age x_0, but that because of some pediatric advance, a proportion $\bar{p}'(x_0)$, greater than $\bar{p}(x_0)$, now survives. Because $\bar{z}$ increases monotonically with $\bar{p}$, $\bar{z}(x_0)$ will increase. Consequently, if the values $\mu(x)$, where x is greater than x_0, remain the same, the values of $\bar{\mu}(x)$, where x is greater than x_0, will increase. However, if observed death rates at younger ages are reduced to low levels, further progress will add fewer and fewer *additional* persons to the ranks of the elderly. Thus, progress in reducing population mortality rates will not be slowed to the extent it previously was.

It follows from equation (8) that

$$\frac{d\bar{\mu}(x)/dx}{\bar{\mu}(x)} = \frac{d\mu(x)/dx}{\mu(x)} + \frac{d\bar{z}(x)/dx}{\bar{z}(x)} \tag{30a}$$

Up until now this report has focused on a single cohort aging through time; thus x represents both age and time. Generalization to the case of multiple cohorts is straightforward: let $\mu(a,y)$, $\bar{\mu}(a,y)$, and $\bar{z}(a,y)$ be the values of μ, $\bar{\mu}$, and $\bar{z}$ for a cohort of age a in year y. Then, fundamental theorem (8) can be rewritten as

$$\bar{\mu}(a,y) = \mu(a,y)\bar{z}(a,y) \tag{8'}$$

and it follows that

$$\frac{\partial\bar{\mu}(a,y)/\partial a}{\bar{\mu}(a,y)} = \frac{\partial\mu(a,y)/\partial a}{\mu(a,y)} + \frac{\partial\bar{z}(a,y)/\partial a}{\bar{z}(a,y)} \tag{30b}$$

and that

$$\frac{\partial\bar{\mu}(a,y)/\partial y}{\bar{\mu}(a,y)} = \frac{\partial\mu(a,y)/\partial y}{\mu(a,y)} + \frac{\partial\bar{z}(a,y)/\partial y}{\bar{z}(a,y)} \tag{30c}$$

Both equations are interesting, but for the purposes of studying the dynamics of mortality progress over time, the second equation is the relevant one.

Let

$$\pi_a(y) = -\frac{\partial\mu(a,y)/\partial y}{\mu(a,y)} \tag{31a}$$

and

$$\bar{\pi}_a(y) = -\frac{\partial\bar{\mu}(a,y)/\partial y}{\bar{\mu}(a,y)} \tag{31b}$$

Thus, π and $\bar{\pi}$ are measures of the rate of progress in reducing individual and population death rates. Equation (30c) can be rewritten as

$$\bar{\pi}_a(y) = \pi_a(y) - \frac{\partial\bar{z}(a,y)/\partial y}{\bar{z}(a,y)} \tag{32}$$

When individuals remain at the same level of relative-risk for life, progress in reducing individual death rates will reduce the value of the negative term in this formula; at any age a the value of $\bar{z}(a,y)$ will approach one as y increases, and the value of $\partial\bar{z}(a,y)/\partial y$ will approach zero. This is easy to see in the special case where relative-risk is gamma distributed at birth with a mean and variance of one. Then, $\bar{z}(x)$ equals $\bar{p}(x)$ so

$$\frac{\partial\bar{z}(a,y)/\partial y}{\bar{z}(a,y)} = \frac{\partial\bar{p}(a,y)/\partial y}{\bar{p}(a,y)} \tag{33}$$

The proportion surviving at any age a will clearly approach one as progress in reducing death rates continues. Furthermore, the change over time in the proportion surviving will approach zero.

Equation (32) consequently indicates that as progress in reducing individual death rates continues,

$$\bar{\pi}_a(y) \to \pi_a(y) \qquad \text{for any } a \tag{34}$$

Since progress in reducing death rates permits frailer individuals to survive to older ages,

$$\partial \bar{z}(a,y)/\partial a < 0 \tag{35}$$

But, of course, $\bar{z}(a,y)$ is greater than zero. Therefore,

$$\bar{\pi}_a(y) < \pi_a(y) \qquad \text{for any } a \tag{36}$$

In short, the observed rate of progress in reducing the population death rate at any age a will be less than, but will approach over time, the rate of progress in reducing individual death rates at age a. Table 5 presents some numerical results concerning $\bar{\pi}_a(y)$ when $\pi_a(y)$ is constant for all a and y; Figure 3 depicts the pattern of these results.

The pattern shown in Figure 3 is roughly the pattern actually observed in the United States, Sweden, and other countries over the course of this century. Thus, the observed acceleration of progress in reducing mortality at older ages may be, at least in part, an artifact of heterogeneity. To the extent this is true, death rates after age 70 and especially after age 80 may decline faster in the future than now predicted – and at an accelerating rate. The various implications of an increase in the size of the elderly populations, including the pressures it would place on pension systems, are discussed by Myers (1981); the work by Arthur (1981) is also relevant.

TABLE 5 The acceleration in observed rates of progress in reducing mortality rates.

	Observed rate of progress $\bar{\pi}(a,y)$ when age a equals:			
Year y	20	40	60	80
0	0.00986	0.00894	0.00528	0.00131
40	0.00991	0.00927	0.00626	0.00184
80	0.00994	0.00950	0.00714	0.00252
120	0.00996	0.00966	0.00788	0.00334
∞	0.01000	0.01000	0.01000	0.01000

NOTE: It is assumed that the rate of progress on the individual level is 0.01;

$$[\partial \mu(a,y)/\partial y]/\mu(a,y) = \pi = 0.01 \qquad \text{for all } a,y$$

Furthermore, z is assumed to be gamma distributed with mean one and variance one at birth, and $\mu(a,0) = 0.0002 \exp(0.1a)$.

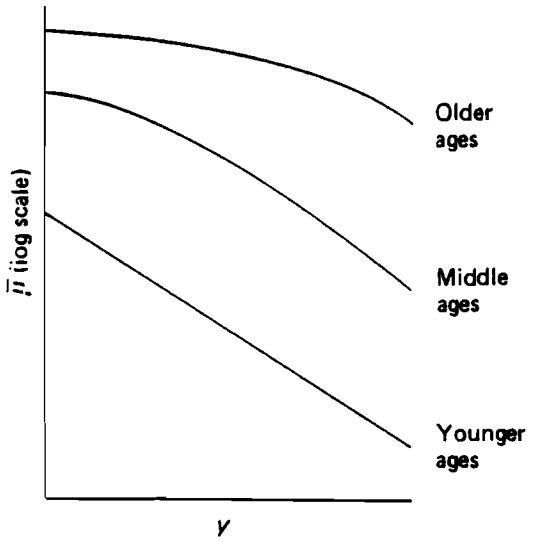

FIGURE 3 Trajectories of progress in reducing mortality rates.

WHEN PROGRESS STOPS

Suppose progress has been made over a number of years in reducing individual mortality rates and then, suddenly, the progress stops so that the mortality rates henceforth remain constant. In the succeeding years (i.e., as y increases), the value of $\bar{p}(a,y)$, the proportion of the original birth cohort surviving to age a in year y, will increase and then level off. The increase in $\bar{p}(a)$ will result from the aging of the younger cohorts that have experienced lower death rates because of the previous progress. Since, as noted earlier, $\bar{z}$ is a monotonically increasing function of $\bar{p}$, it follows that $\bar{z}$ will increase as well. The value of $\mu(a,y)$, any a and y, will be constant – that is what no progress means. But

$$\bar{\mu}(a,y) = \mu(a,y)\bar{z}(a,y) \tag{8'}$$

Thus, $\bar{\mu}(a,y)$ at any age a will increase over time.

In short, current mortality rates for populations are lower than the mortality rates that would prevail if current mortality rates for individuals persisted. If progress in health conditions stops, death rates will rise. This implies that estimates of "current life expectancy" are too high. These estimates are based on current population death rates, but they are supposed to represent what life expectancy would be if health conditions remained unchanged. Vaupel *et al.* (1979a) indicate how the correct value of current life expectancy, adjusted for the effects of heterogeneity and past health progress, might be calculated. Table 6 and Figure 4 compare the patterns of $\mu(a,y)$ and $\bar{\mu}(a,y)$ when health progress stops.

If progress in reducing μ accelerates and decelerates over time, the observed trajectory of $\bar{\mu}$ will be bumpy and might show periods of apparent negative progress; this phenomenon might underlie the increase in death rates observed in the United States in the middle and late 1960s, following a relatively rapid decrease in the 1950s.

TABLE 6 When progress in reducing mortality rates stops.

Year y	$\mu(60,y)$	$\bar{\mu}(60,y)$
0	0.08069	0.04264
20	0.06606	0.03817
40	0.05409	0.03384
60	0.04428	0.02972
80	0.03625	0.02588
81	0.03625	0.02595
90	0.03625	0.02635
∞	0.03625	0.02662

ASSUMPTIONS: $\mu(a,0) = 0.0002 \exp(0.1a)$
$\mu(a,y) = \mu(a,0) \exp(-0.01y), y \leq 80$
$\mu(a,y) = \mu(a,80), y > 80$

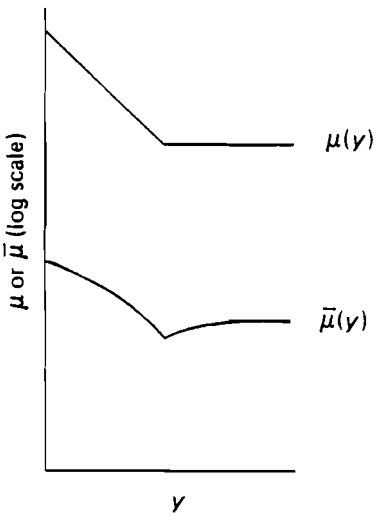

FIGURE 4 When progress in reducing mortality rates stops.

INDEPENDENT COMPETING RISKS

Suppose there are several causes of death and that an individual can be at different relative-risks for the different causes. Let z_i denote the level of relative-risk for cause of death i and let $\mu_i(x,z_i)$ be the death rate from cause i at time (or age) x for individuals at relative-risk z_i. As before, define z_i such that

$$\mu_i(x, z_i) = z_i \mu_i(x, 1) = z_i \mu_i(x) \tag{4'}$$

Assume that an individual's relative-risk for any cause of death is independent of his or her relative-risk for any other cause of death. Then, as shown in the Appendix, a straightforward generalization of fundmental theorem (8) yields

$$\bar{\mu}_i(x) = \mu_i(x) \bar{z}_i(x) \tag{8'a}$$

and

$$\bar{\mu}(x) = \sum_{i=1}^{n} \bar{\mu}_i(x) \tag{8'b}$$

where $\bar{\mu}_i$ represents the population death rate from cause i and where $\bar{z}_i(x)$ is the mean relative-risk from cause i among the individuals surviving to time x. The value of $\bar{z}_i(x)$ for any cause of death i can be calculated on the basis of $f_0(z_i)$, the distribution of z_i at birth, and $\mu_i(x)$, the death rate from cause i:

$$\bar{z}_i(x) = \frac{\int_0^\infty z_i f_0(z_i) \exp\left[-\int_0^x z_i \mu_i(t)\, dt\right] dz_i}{\int_0^\infty f_0(z_i) \exp\left[-\int_0^x z_i \mu_i(t)\, dt\right] dz_i} \tag{13'}$$

Thus, the dynamics of mortality from any specific cause of death can be studied without knowing the death rates and distributions of relative-risks for other causes of death.

Suppose that the z_i are gamma distributed with mean one and variances σ_i^2. (As before, the means might as well be set equal to one, as in that case the "standard" individual at relative-risk one will be the mean individual at birth.) Then equation (19) generalizes to

$$\bar{z}_i(x) = 1/[1 + \sigma_i^2 H_i(x)] \tag{19'}$$

where

$$H_i(x) = \int_0^x \mu_i(t)\, dt \tag{37}$$

Furthermore, equation (18) generalizes to

$$\bar{z}_i(x) = \bar{p}_i(x)^{\sigma_i^2} \tag{18''}$$

where $\bar{p}_i(x)$ is the proportion of the population that would survive to age x if i were the only cause of death:

$$\bar{p}_i(x) = \exp\left[-\int_0^x \bar{\mu}_i(t)\,dt\right] \tag{38}$$

The formulas for the uniform distribution (23) and the two-point distribution (22) similarly generalize.

Thus, the case of independent, competing risks is almost as easy to analyze as the simpler case of a single cause of death. In a sense, the competing risk case adds another layer or dimension of heterogeneity as now individuals not only differ from each other, but they also differ within themselves in susceptibility to various causes of death.

Patterns of aging for individuals can be compared with observed patterns of aging for the surviving cohort in much the same way when there are several causes of death as when there is only a single cause of death. Figure 5 presents an example. The mortality curve shown in Figure 5, which is plotted on a log scale, is intriguing because it resembles the observed mortality curves of most developed countries: mortality falls off after infancy, begins increasing again after age 7 or so, rises through a hump roughly between

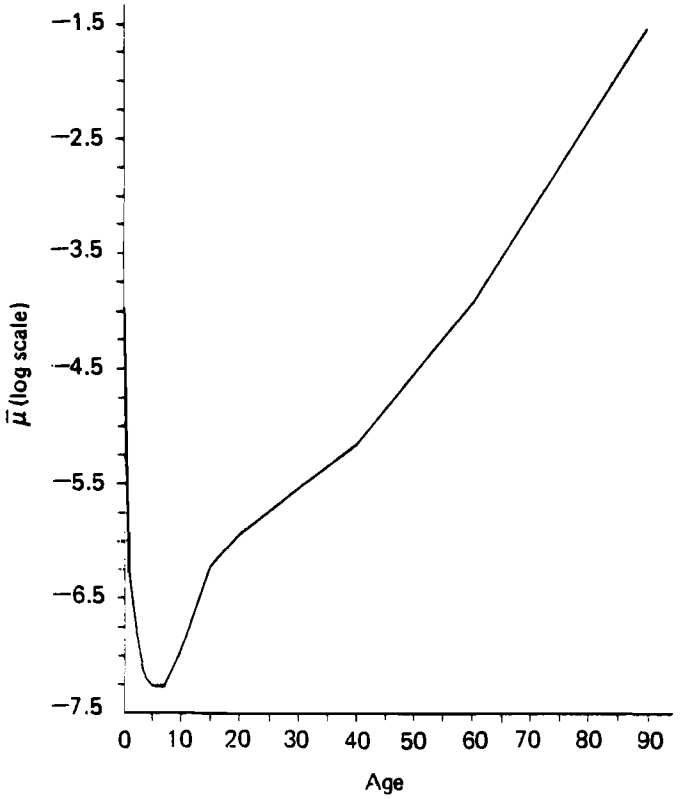

FIGURE 5 A population mortality curve produced by three causes of death. The three independent causes of death act, on the individual level, as follows: $\mu_1(x) = 0.02$ and z_1 is gamma distributed with $\sigma_1^2 = 500$; $\mu_2(x) = 0.00001 \exp(0.4x)$ and z_2 is gamma distributed with $\sigma_2^2 = 200$; $\mu_3(x) = a \exp(bx) \exp\{a[\exp(bx) - 1]/b\sigma_3^2\}$, $a = 0.00015$, $b = 0.08$, and z_3 is gamma distributed with $\sigma_3^2 = 1$.

ages 15 and 30, and then at older ages increases more or less exponentially. Figure 5 was generated by assuming there were three causes of death. For individuals, the incidence of the first cause is constant, the incidence of the second cause increases exponentially, and the incidence of the third cause increases according to the double-exponential form that produces, on the population level, an observed exponential increase.

Just as mortality convergences and crossovers for two populations may be artifacts of heterogeneity, convergences and crossovers for two causes of death may also be artifacts of heterogeneity. In the earlier discussion of population crossovers, the subscript i denoted population 1 or 2 – e.g., $\bar{\mu}_i$ was the death rate for population i. The mathematics is equally valid if the subscript i denotes cause of death 1 or 2. So, for example, cause of death 2 might be twice as likely as cause of death 1, at all ages, for all individuals. If the variance in z_2, however, is greater than twice the variance in z_1, then the observed rate of death from cause 2 in the surviving cohort will approach and eventually fall below the observed rate for cause 1.

What will be the effect of progress in reducing individual death rates on observed progress in reducing deaths in surviving cohorts? For any specific cause of death, the mathematics will be the same as outlined in the section on progress above. Furthermore, in the case being considered here of independent causes of death, progress in reducing one cause of death will have no effect on $\mu_i(x)$ or $\bar{\mu}_i(x)$ for any other cause of death i. Since everyone has to die of something, the *number* of people eventually dying from other causes will increase, but the death *rates* μ_i and $\bar{\mu}_i$ will not change.

CORRELATED CAUSES OF DEATH

When causes of death are not independent but are correlated with each other, the mathematics becomes more complicated. The fundamental equations

$$\bar{\mu}_i(x) = \mu_i(x)\bar{z}_i(x) \tag{8'a}$$

and

$$\bar{\mu}(x) = \sum_{i=1}^{n} \bar{\mu}_i(x) \tag{8'b}$$

are still valid, but now the value of $\bar{z}_i(x)$ depends on the death rates and distributions of relative-risks for correlated causes of death:

$$\bar{z}_i(x) = \frac{\int_0^\infty \cdots \int_0^\infty z_i f_0(z_1,\ldots,z_n) \exp\left[-z_1H_1(x) - \cdots - z_nH_n(x)\right] \mathrm{d}z_1,\ldots,\mathrm{d}z_n}{\int_0^\infty \cdots \int_0^\infty f_0(z_1,\ldots,z_n) \exp\left[-z_1H_1(x) - \cdots - z_nH_n(x)\right] \mathrm{d}z_1,\ldots,\mathrm{d}z_n} \tag{39}$$

where, as before,

$$H_i(x) = \int_0^x \mu_i(t)\,dt$$

As a simple example, consider the following special case. Suppose that there are two causes of death and that, as in the mover/stayer model, there are two kinds of people. Let $\mu_1(x)$ and $\mu_2(x)$ be the death rates from cause 1 and 2 for the standard individual in the first group, and let $\mu_1'(x)$ and $\mu_2'(x)$ be the rates for the second group. Finally, suppose the rates are interrelated as follows:

$$0 < \mu_1'(x) < \mu_1(x) \qquad \text{for all } x \tag{40a}$$

and

$$\mu_2'(x) = 0 \qquad \text{for all } x \tag{40b}$$

Thus, the second, "robust" group does not die from cause 2 and faces a lower death rate than the first group does from cause 1.

Let $\pi(x)$ denote the proportion of the total population that is in the first group at time x. The observed death rate for the first cause of death will be

$$\bar{\mu}_1(x) = \pi(x)\mu_1(x) + [1 - \pi(x)]\,\mu_1'(x) \tag{41a}$$

and the observed death rate for the second cause of death will simply be

$$\bar{\mu}_2(x) = \pi(x)\mu_2(x) \tag{41b}$$

Suppose some progress is made in reducing the incidence of the second cause of death. Then the observed death rate from the first cause will increase. This observed death rate is the weighted average of the death rates for the first and second groups. If death rates for the first group are reduced (as a result of progress against the second cause of death), more of this group will survive. The value of $\pi(x)$ will increase and since $\mu_1(x)$ exceeds $\mu_1'(x)$, the value of $\bar{\mu}_1(x)$ will also increase. The value of $\pi(x)$, by the way, is given by

$$\pi(x) = \frac{\pi(0)\exp\left\{-\int_0^x [\mu_1(t) + \mu_2(t)]\,dt\right\}}{\pi(0)\exp\left\{-\int_0^x [\mu_1(t) + \mu_2(t)]\,dt\right\} + [1 - \pi(0)]\exp\left[-\int_0^x \mu_1'(t)\,dt\right]} \tag{42}$$

A more general situation in which causes of death are correlated can be described as follows. Let $z_0, \ldots, z_n$ be independent relative-risks with mean one. Let the death rate for an individual be given by

$$\mu_i(x,z) = [w_i z_0 + (1 - w_i) z_i]\,\mu_i(x) \tag{43}$$

where z is the vector of relative-risks for the individual and w_i is a weight such that

$$0 \leqslant w_i \leqslant 1 \qquad i = 1,\ldots,n \tag{44}$$

The basic idea is that an individual's risk from any specific cause of death i depends on a general relative-risk (or frailty) factor z_0 and a specific relative-risk factor z_i.

It can be readily shown that

$$\bar{\mu}_i(x) = [w_i \bar{z}_0(x) + (1 - w_i)\bar{z}_i(x)]\, \mu_i(x) \tag{45}$$

If the z_i are gamma distributed with mean one and variances σ_i^2, then

$$\bar{z}_0(x) = \frac{1}{1 + \sigma_0^2 \sum_{i=1}^{n} w_i H_i(x)} \tag{46a}$$

and

$$\bar{z}_i(x) = \frac{1}{1 + \sigma_i^2 (1 - w_i) H_i(x)} \qquad i = 1,\ldots,n \tag{46b}$$

If w_j is greater than zero, then reducing the incidence of cause of death j will increase $\bar{z}_0(x)$. This increase in $\bar{z}_0(x)$ will, if w_i is greater than zero, result in an increase in the observed incidence of cause of death i. Indeed, if $H_j(x)$ is reduced by δ_j, then $\bar{\mu}_i(x)$ will increase by

$$\frac{\delta_j w_j w_i \mu_i(x)}{\left[1 + \sigma_0^2 \sum_{k=1}^{n} H_k(x) w_k\right]\left[1 + \sigma_0^2 \sum_{k=1}^{n} H_k(x) w_k - \delta_j w_j\right]} \tag{47}$$

In short, when relative-risks from different causes of death are positively correlated, progress against one cause of death may lead to observed increases in the rates of other causes of death.

WHEN THE RELATIVE-RISKS OF INDIVIDUALS CHANGE PROPORTIONATELY OVER TIME

So far it has been assumed that an individual is born at some level of relative-risk and remains at that level for life. Clearly, however, individuals' relative-risk levels may in some situations change significantly over time. Sometimes this change is caused by factors, such as improvements in living conditions or progress in medical technology, that may affect individuals proportionately to their current relative-risk levels. That is, for all individuals,

$$dz(x)/dx = -\varphi(x) z(x) \;, \qquad z(0) = z_0 \tag{48a}$$

where $z(x)$ is an individual's relative-risk at time x, and $\varphi(x)$ measures the intensity of the change. Alternatively, the value of $z(x)$ could be given by

$$z(x) = g(x) z_0 \tag{48b}$$

where z_0 is an individual's relative-risk at birth and $g(x)$ measures the cumulative change. The values of $\varphi(x)$ and $g(x)$ are related by

$$g(x) = \exp\left[-\int_0^x \varphi(t)\,dt\right] \tag{49}$$

Because $\mu(x,z)$ equals $z\mu(x)$, it follows that

$$\mu[x, z(x)] = z_0 g(x) \mu(x) \tag{50}$$

Let

$$\mu'(x) = g(x)\mu(x) \tag{51}$$

The function $\mu'(x)$ can be interpreted as describing the trajectory of death rates for the standard individual under the changing conditions described by $g(x)$. Then, the fundamental equation becomes

$$\bar{\mu}(x) = \mu'(x)\bar{z}'(x) \tag{8''}$$

where, analogously to previous formulas,

$$\bar{z}'(x) = \frac{\int_0^\infty z f_0(z) \exp\left[-\int_0^x \mu'(t)\,dt\right] dz}{\int_0^\infty z f_0(z) \exp\left[-\int_0^x \mu'(t)\,dt\right]} \tag{13''}$$

In short, by combining the function $g(x)$ with $\mu(x)$, all the mathematical apparatus derived earlier can still be applied.

As shown in the Appendix, $g(x)$ could describe a stochastic process. After a particular realization of $g(x)$ is known, then the equations above would hold. Before $g(x)$ is known, the equations hold for expected values; if

$$\mu'(x) = \bar{g}(x)\mu(x) \tag{51'}$$

where $\bar{g}(x)$ is the conditional expectation of $g(x)$ as defined in the Appendix, and if z and $g(x)$ are independent, then the expected mortality curve $\bar{\bar{\mu}}(x)$ is given by

$$\bar{\bar{\mu}}(x) = \mu'(x)\bar{z}'(x) \tag{8'''}$$

where $\bar{z}'(x)$ is given, as before, by equation (13) and where $\bar{\bar{\mu}}(x)$ may be considered a conditional expectation of the observed mortality rate $\bar{\mu}(x)$, as discussed in the Appendix.

DEATH AND DEBILITATION

In some situations death may be associated with some illness, such as tuberculosis or rheumatic fever, or some catastrophe that not only kills people but that also weakens the survivors. To model this kind of correlation between death and debilitation, suppose:

$$z(x) = z_0[1 + \alpha H(x)] \tag{52}$$

for all individuals in the population. Thus, the greater the cumulative death rate $H(x)$ has been, the frailer each of the surviving individuals will be.

Since equation (52) is just a special case of equation (48b), equations (51), (8″), and (13″) can be used to analyze this situation. For illustrative purposes, it is sufficient to consider a simple, concrete instance. Suppose, for example, that z_0 is gamma distributed with mean one and variance σ^2. And suppose that $\mu(x)$ is constant and equals c at all ages x. Then,

$$\bar{\mu}(x) = \frac{c + \alpha c^2 x}{1 + \sigma^2 cx + \sigma^2 \alpha c^2 x^2/2} \tag{53}$$

If the debilitating effect is small relative to the selection effect of heterogeneity – specifically, if α is less than or equal to σ^2 – then $\bar{\mu}(x)$ will decline with age and approach zero. On the other hand, if α exceeds σ^2, then $\bar{\mu}(x)$ will initially rise above the level c, but will then start to decline, will fall below c when

$$x = (2/c)(x - \sigma^2)/(\alpha\sigma^2) \tag{54}$$

and will eventually approach zero. Thus, if α is big enough, the debilitation effect will dominate for a few years until the selection effect of heterogeneity takes over.

A RANDOM WALK THROUGH RELATIVE-RISK

Factors such as further education, increasing income, decreasing alcohol consumption, increasing cigarette consumption, and other changes in life style, living conditions, work environment, and so on may gradually alter any particular individual's relative-risk (or frailty) level relative to other individuals' levels. Suppose that the process is the usual kind of random walk known as a Wiener or Brownian-motion process. In this kind of process, the change in an individual's relative-risk at any instant in time is proportional to the individual's level of relative-risk. Furthermore, the cumulative change over an interval of time is proportional to the length of the interval. More exactly,

$$dz(t) = z(t)b(t)\, dw(t) \ , \qquad z(0) = z_0 \tag{55}$$

where $w(t)$ is a Wiener process conditionally independent of z_0 when time of death exceeds t and $b(t)$ is some deterministic function such that

$$\int_0^\infty b^2(t)\ \mathrm{d}t < \infty \tag{56}$$

As shown in the Appendix, if T denotes time of death, then

$$\bar{\bar{\mu}}(x) = \mu(x)\bar{z}(x)E\left\{\exp\left[\int_0^x b(s)\ \mathrm{d}w(s) - \tfrac{1}{2}\int_0^x b(s)\ \mathrm{d}s\right] \mid T > x\right\} \tag{8''''}$$

where $\bar{z}(x)$ is defined, as before, by equation (13). Thus, remarkably, the mathematical apparatus developed above for the special case of unchanging individual relative-risks also holds, in terms of expected observed mortality $\bar{\bar{\mu}}(x)$, for the more general case where the relative-risk level of each individual is gradually changing according to a random walk process. However, the calculation of the conditional mathematical expectation on the right-hand side of equation (8'''') requires more sophisticated methods of estimation based, for example, on the theory of random point processes (Yashin 1970, 1978; Snyder 1975; Brémaud 1981).

The three kinds of change in relative-risk discussed above – deterministic proportional change for all individuals, stochastic proportional change for all individuals, and independent random walks for each individual – can be combined with obvious changes in the mathematics.

CONCLUSION

"Individuals", whether people, plants, animals, or machines, differ from one another. Sometimes the differences affect the probability of some major transition, such as dying, moving, marrying, or converting. If so, the observed dynamics of the behavior of the surviving population – the population that has not yet made the transition – will systematically deviate from the dynamics of the behavior of any of the individuals that make up the population. Most of the examples and terminology of this report were drawn from the study of human mortality, but the mathematics can be applied to various kinds of heterogeneous populations for such purposes as explaining population patterns, making inferences about individual behavior, and predicting or evaluating the impact of alternative control mechanisms, policies, and interventions.

Among the interesting results discussed in this study are:

- Individuals age faster than heterogeneous cohorts.
- Observed mortality convergences and crossovers, both between populations and between causes of death, may be artifacts of heterogeneity.
- Progress in reducing mortality at younger ages or from some causes of death may increase observed mortality at older ages or from other causes of death.

- Slow but accelerating rates of mortality progress in old age may be an artifact of heterogeneity, with a significant consequence: the elderly population may be substantially larger in the future than currently predicted.

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APPENDIX

A1. Proof of Equation (6)

Let $f(z)$ be the probability density function of frailty z and let T be the random death time. Denote by $\varphi(t\,|\,z)$ the conditional probability density of death time T when frailty z is given. Note that

$$\varphi(t\,|\,z) = z\mu(t) \exp\left[-z \int_0^t \mu(x)\ \mathrm{d}x\right]$$

where $\mu(x)$ is the age-specific death rate for the standard individual with frailty z equaling one. Using the notation $g(t,z)$ for the joint probability distribution function of death time T and frailty z we get, multiplying $f(z)$ and $\varphi(t\,|\,z)$,

$$g(t,z) = f(z)\varphi(t\,|\,z)$$

According to the definition of $\bar{\mu}(x)$

$$\bar{\mu}(x) = \frac{h(x)}{P(T>x)}$$

where $h(x)$ is the probability density function for death time T.

Note that

$$h(x) = \int_0^\infty g(x,z)\ \mathrm{d}z = \int_0^\infty f(z)\varphi(x\,|\,z)\ \mathrm{d}z$$

Using the expression for $\varphi(t\,|\,z)$ we have for $\bar{\mu}(x)$

$$\bar{\mu}(x) = \frac{\int_0^\infty z\mu(x) \exp\left[-z \int_0^x \mu(t)\ \mathrm{d}t\right] f(z)\ \mathrm{d}z}{P(T>x)}$$

Noting that according to the formula for $\varphi(t\,|\,z)$

$$P(T>x\,|\,z) = \exp\left[-z \int_0^x \mu(t)\ \mathrm{d}t\right]$$

the formula for $\bar{\mu}(x)$ may be rewritten as follows

$$\bar{\mu}(x) = \frac{\mu(x) \int_0^\infty P(T > x \mid z) f(z)\, \mathrm{d}z}{P(T > x)}$$

Denoting by $f_x(z)$ the conditional probability density function of z when event $\{T > x\}$ is given and noting that according to Bayes formula

$$f_x(z) = \frac{P(T > x \mid z) f(z)}{P(T > x)}$$

we have for $\bar{\mu}(x)$

$$\bar{\mu}(x) = \mu(x) \int_0^\infty z f_x(z)\, \mathrm{d}z = \mu(x) E(z \mid T > x)$$

completing the proof.

A2. The Competing Risk Case

Let frailty be the vector $z = (z_1, z_2, \ldots, z_n)$. Denote by T_i the random death times caused by frailty z_i, where $i = 1, 2, \ldots, n$, and let $T = \min\{T_i, i = 1, 2, \ldots, n\}$. Let the density function of T when frailty z is given be

$$\varphi(t \mid z) = \left[\sum_{i=1}^n z_i \mu_i(t)\right] \exp\left[-\sum_{i=1}^n z_i \int_0^t \mu_i(x)\, \mathrm{d}x\right]$$

Note that from this formula it follows that

$$P(T > x \mid z) = \exp\left[-\sum_{i=1}^n z_i \int_0^x \mu_i(t)\, \mathrm{d}t\right]$$

As in the scalar case note that

$$\bar{\mu}(x) = \frac{\mathrm{d}P(T \leqslant x)/\mathrm{d}x}{P(T > x)}$$

Denoting by $f(z)$ the density probability function of vector $z = (z_1, \ldots, z_n)$, we have

$$\bar{\mu}(x) = \frac{\int_0^\infty [\mathrm{d}P(T \leqslant x \mid z)/\mathrm{d}x] f(z)\, \mathrm{d}z}{P(T > x)}$$

or using the formula for $\varphi(t \mid z)$

$$\bar{\mu}(x) = \frac{\int_0^\infty \left[\sum_i z_i \mu_i(x)\right] \exp\left[-\sum_{i=1}^n z_i \int_0^x \mu_i(t)\, dt\right] f(z)\, dz}{P(T>x)}$$

Noting that

$$\frac{\exp\left[-\sum_{i=1}^n z_i \int_0^x \mu_i(t)\, dt\right] f(z)}{P(T>x)} = f_x(z)$$

where $f_x(z)$ is the conditional probability density function of vector frailty $z = (z_1, \ldots, z_n)$ when the event $\{T>x\}$ is given, we get for $\bar{\mu}(x)$

$$\bar{\mu}(x) = \sum \mu_i(x) \hat{z}_i(x)$$

where

$$\hat{z}_i(x) = E\{z_i \mid T>x\}$$

It is very important to know when $\hat{z}_i(x)$ coincides with $\bar{z}_i(x)$, where $\bar{z}_i = E\{z_i \mid T_i > x\}$ is the conditional frailty that was defined before. For this purpose note that the random event $\{T>x\}$ may be represented as

$$\{T>x\} = \bigcap_{i=1}^n \{T_i > x\}$$

The equality $\hat{z}_i(x) = \bar{z}_i(x)$ means that

$$E(z_i \mid \bigcap_i \{T_i > x\}) = E(z_i \mid T_i > x)$$

The last equality may take place only in the case when frailty z_i for any i does not depend on T_j, where $j \neq i$, and $i,j = 1,2,\ldots,n$.

A3. The Proof of the Formula for $\bar{\bar{\mu}}(x)$

Assume that the following representation for the age-specific mortality rate $\mu(x,z)$ is valid

$$\mu(x,z) = zg(x)\mu(x)$$

where $g(x)$ is some integrable random function that is independent of z and takes values on the real line. According to the definition of $\bar{\bar{\mu}}(x)$

$$\bar{\bar{\mu}}(x) = -\frac{dP(T>x)/dx}{P(T>x)}$$

Let the symbol E_Q denote the operation of averaging with respect to measure Q, which is defined in the space of functions $g(x)$. Then for $\bar{\bar{\mu}}(x)$ we can write

$$\bar{\bar{\mu}}(x) = \mu(x)\frac{E_Q \int_0^\infty zg(x)\exp\left[-z\int_0^x g(s)\mu(s)\,ds\right] f(z)\,dz}{E_Q \int_0^\infty \exp\left[-z\int_0^x g(s)\mu(s)\,ds\right] f(z)\,dz}$$

where $f(z)$ is the probability density function of z.

It is not difficult to see that

$$\bar{\bar{\mu}}(x) = \mu(x)E(zg(x)|\,T>x)$$

Since variables z and $g(x)$ are conditionally independent, the formula for $\bar{\bar{\mu}}(x)$ may be rewritten as follows:

$$\bar{\bar{\mu}}(x) = \mu(x)E(z|\,T>x)E[g(x)|\,T>x]$$

or using the previous notation

$$\bar{\bar{\mu}}(x) = \mu(x)\bar{z}(x)\bar{g}(x)$$

A4. Frailty as a Solution of Stochastic Differential Equations

Assume that frailty $z(t)$ is governed by the following stochastic differential equation

$$dz(t) = z(t)b(t)\,dw(t)\ , \quad z(0)>0$$

where $z(0)$ does not depend on $w(t)$ and

$$\int_0^t b^2(t)\,dt < \infty$$

The solution of this equation may be found in the following way. Apply the stochastic differentiation formula (Ito formula) to the function $y(t) = \ln z(t)$ (Liptzer and Shirjaev 1977), which yields

$$y(t) = y(0) + \int_0^t b(s)\,dw(s) - (\tfrac{1}{2})\int_0^t b^2(s)\,ds$$

and consequently for $z(t)$

$$z(t) = z(0) \exp\left[\int_0^t b(s)\ \mathrm{d}w(s) - (\tfrac{1}{2}) \int_0^t b^2(s)\ \mathrm{d}s\right]$$

Denoting by $g(t) = \exp\left[\int_0^t b(s)\ \mathrm{d}w(s) - (\tfrac{1}{2}) \int_0^t b^2(s)\ \mathrm{d}s\right]$ and recalling that $\mu(x,z) = z(0)g(x)\mu(x)$, it follows from section A3 that

$$\bar{\bar{\mu}}(x) = \bar{z}(x)\bar{g}(x)\mu(x)$$

where

$$\bar{g}(x) = E[g(x) \mid T > x]$$

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